# Homework 4
# MTH 869 Algebraic Topology

Joshua Ruiter

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**Proposition 0.1** (Exercise 1.2.6)**.** *Let $S$ be a closed, discrete subset of $\mathbb{R}^n$ with $n \geq 3$. Then $\mathbb{R}^n \setminus S$ is simply connected.*

*Proof.* Let $S = \{s_i\}_{i \in I}$ where $I$ is some arbitrary indexing set. Since $S$ is discrete, for each $i$ there exists $\epsilon_i > 0$ so that $B(s_i, 2\epsilon_i) \cap S = \{s_i\}$. For each $i$, choose $x_i$ to be a point of distance $\epsilon_i$ from $s_i$. We define a zero skeleton $X^0 = \{x_i\}_{i \in I}$.

To each $x_i$, attach an $(n-1)$ cell via a constant map, so that the resulting copy of $S^{n-1}$ is $\partial B(s_i, \epsilon_i)$. That is, the copy of $S^{n-1}$ encloses $s_i$. This forms an $(n-1)$ skeleton. Now attach a single $n$-cell to fill up the rest of the ambient $\mathbb{R}^n$, attaching pieces to each $S^{n-1}$ as necessary. The resulting $n$-dimensional CW complex $X$ is $\mathbb{R}^n \setminus \bigcup_i B(s_i, \epsilon_i)$. Then we see that $X$ is a deformation retract of $\mathbb{R}^n \setminus S$, since we can linearly retract each $B(s_i, \epsilon_i) \setminus \{s_i\}$ to its boundary.

Since $n \geq 3$, we have $n - 1 \geq 2$, so $\pi_1(S^{n-1}) = 0$, and since $X^{n-1}$ is a disjoint union of $n-1$ spheres, thus $\pi_1(X^{n-1}, x_0) = 0$. Now by Proposition 1.26(b), since $X$ is obtained from $X^{n-1}$ by attaching $n$-cells, $\pi_1(X) \cong \pi_1(X^{n-1}) = 0$. Thus $X$ is simply connected, so $\mathbb{R}^n \setminus S$, which is homotopy equivalent to $X$, is also simply connected. □

**Proposition 0.2** (Exercise 1.2.7)**.** *Let $X$ be the quotient space of $S^2$ formed by identifying the north and south poles. Then $X$ is a CW complex and has fundamental group $\pi_1(X) \cong \mathbb{Z}$.*

*Proof.* We take the zero-skeleton $X^0$ to be a single point $x_0$, and form the one-skeleton $X^1$ by attaching a single interval and identifying both endpoints with $x_0$. Thus, $X^1$ is homeomorphic to $S^1$. Then we define an attaching map $\phi : S^1 \to X^1 \cong S^1$ as follows. We view $S^1$ as a square, labelling the sides $a, b, c, d$ in clockwise order. We send $a$ to wrap once around $X^1$ clockwise. We send $b$ and $d$ both to the basepoint $x_0$. We send $c$ to wrap once around $X^1$ counterclockwise. Thus we attach a 2-cell to $S^1$. The resulting space is $X$.

Now we can compute $\pi_1(X)$. By proposition 1.26(a), the inclusion $X^1 \hookrightarrow X$ induces a surjection $\iota_* : \pi_(X^1, x_0) \to \pi_1(X, x_0)$, where the kernel is generated by conjugations of $\phi$ by change of basepoint paths. Since $\phi$ has the right basepoint, the kernel is generated by $[\phi]$. But $[\phi] = 0$ in $\pi_1(X^1, x_0)$. Thus $\pi_1(X, x_0) \cong \pi_1(X^1, x_0) \cong \mathbb{Z}$. □

**Lemma 0.3** (for Exercise 1.2.9)**.** *Let $M_g$ be the closed orientable surface of genus $g$, and let $M_g'$ be the closed surface obtained by removing an open disc from $M_g$. Then $M_g' \simeq \bigvee_{i=1}^{2g} S^1$, so $\pi_1(M_g')$ is the free group on $2g$ generators.*

*Proof.* We can view $M_g$ as a regular polygon with $4g$ sides with sides identified in a particular way. That is, going clockwise around we label the sides

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_{2g}, b_{2g}, a_{2g}^{-1}, b_{2g}^{-1}$$

and make the implied identifications. Then we view $M_g'$ as this same space, with a small open disk removed from the center of the polygon. Thinking of our polygon as centered at the origin in the plane, and our removed open disk as centered at the origin, notice that each radial line connecting the central disk and the boundary the polygon is contractible. Thus, $M_g'$ is homotopic to just the polygon with sides identified. This polygon is a wedge sum of $2g$ circles. Thus $M_g' \simeq \bigvee_{i=1}^{2g} S^1$. We already computed $\pi_1\left(\bigvee_{i=1}^{2g} S^1\right)$ to be the free group on $2g$ generators, using Van Kampen's theorem. □

**Lemma 0.4** (for Exercise 1.2.9)**.** *Abelianization of groups is a covariant functor.*

*Proof.* We have a rule ${}^{\text{ab}} : \text{Grp} \to \text{Ab}$ that sends a group $G$ to the abelian group $G^{\text{ab}} = G/[G,G]$. Given a group homomorphism $f : G \to H$, we define $f^{\text{ab}} : G^{\text{ab}} \to H^{\text{ab}}$ by $x[G,G] \to f(x)[H,H]$. First we check that $f^{\text{ab}}$ is well defined. Suppose $x[G,G] = y[G,G]$. Note that $f([G,G]) \subset [H,H]$ since $f$ maps commutators to commutators, as it is a homomorphism.

$$xy^{-1} \in [G,G] \implies f(xy^{-1}) \in [H,H] \implies f(x)f(y)^{-1} \in [H,H] \implies f(x)[H,H] = f(y)[H,H]$$

Thus $f^{\text{ab}}$ is well defined. Now we show that it is a covariant functor. First, $\text{Id}_G^{\text{ab}} : G^{\text{ab}} \to G^{\text{ab}}$ is the identity on $G^{\text{ab}}$, as can be seen from the definition of $f^{\text{ab}}$. Let $f : G \to H$ and $g : H \to K$ be group homomorphisms. Then $(g \circ f)^{\text{ab}} : G^{\text{ab}} \to K^{\text{ab}}$ acts on $x[G,G] \in G^{\text{ab}}$ by

$$(g \circ f)^{\text{ab}}(x[G,G]) = (g \circ f)(x)[K,K] = g(f(x))[K,k] = g^{\text{ab}}(f(x)[H,H]) = g^{\text{ab}} \circ f^{\text{ab}}(x[G,G])$$

Thus $(g \circ f)^{\text{ab}} = g^{\text{ab}} \circ f^{\text{ab}}$, so ${}^{\text{ab}}$ is a covariant functor. □

**Lemma 0.5** (for Exercise 1.2.9)**.** *Let $\phi : G \to H$ and $\psi : H \to G$ be group homomorphisms so that $\phi \circ \psi = \text{Id}_H$. Then $\psi^{\text{ab}}$ is injective.*

*Proof.* By functoriality of ${}^{\text{ab}}$, we have $(\phi \circ \psi)^{\text{ab}} = \phi^{\text{ab}} \circ \psi^{\text{ab}}$ and $(\text{Id}_H)^{\text{ab}} = \text{Id}_{H^{\text{ab}}}$. Thus

$$\phi \circ \psi = \text{Id}_H \implies \phi^{\text{ab}} \circ \psi^{\text{ab}} = \text{Id}_{H^{\text{ab}}}$$

Thus $\psi^{\text{ab}}$ has a left inverse, so it is injective. □

**Proposition 0.6** (Exercise 1.2.9, part one)**.** *Let $M_g$ be the orientable surface of genus $g$, and let $C$ be a circle that separates $M_g$ into two compact subsurfaces $M_h'$ and $M_k'$ obtained from the closed surfaces $M_h$ and $M_k$ by deleting an open disk from each. Then $M_h'$ does not retract onto its boundary circle $C$, and hence $M_g$ does not retract onto $C$.*

*Proof.* Suppose there is a retraction $r : M'_h \to C$. Then the inclusion $\iota : C \hookrightarrow M'_h$ induces an injective homomorphism $\iota_* : \pi_1(C) \to \pi_1(M'_h)$. Note that $\pi_1(C) \cong \mathbb{Z}$ and $\pi_1(M'_h)$ is free on $2h$ generators, which we denote by $F_{2h}$. Thinking of $M'_h$ as a $4g$-sided polygon with a central disk removed, we see that $C$, the boundary of the removed disk, is homotopic to the boundary edge word

$$[a_1, b_1] \dots [a_h, b_h] = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_h b_h a_h^{-1} b_h^{-1}$$

Thus $\iota_*$ maps a generator of $\pi_1(C)$ to the above word. Then applying the $^{\text{ab}}$ functor, we get $\iota_*^{\text{ab}} : \pi_1(C)^{\text{ab}} \to \pi_1(M'_h)^{\text{ab}}$. Since $\pi_1(C)$ is abelian, we drop the $^{\text{ab}}$ and write $\iota_*^{\text{ab}} : \pi_1(C) \to \pi_1(M'_h)^{\text{ab}}$. Note that we have homomorphisms $r_* : \pi_1(M'_h) \to \pi_1(C)$ and $\iota_* : \pi_1(C) \to \pi_1(M'_h)$ so that $r_* \circ \iota_* = \text{Id}_{\pi_1(C)}$. Then by our Lemma 0.5, $\iota_*^{\text{ab}}$ is injective. However, we also compute directly that

$$\begin{aligned} \ker \iota_*^{\text{ab}} &= \{x \in \pi_1(C) : \iota_*^{\text{ab}}(x) = 0\} \\ &= \{x \in \pi_1(C) : \iota_*(x)[F_{2h}, F_{2h}] = 0\} \\ &= \{x \in \pi_1(C) : \iota_*(x) \in [F_{2h}, F_{2h}]\} \\ &= \pi_1(C) \end{aligned}$$

since $\iota_*$ maps a generator for $\pi_1(C)$ to a product of commutators. This is a contradiction, since $\iota_*$ cannot be both injective and have nontrivial kernel. Hence no such retraction exists.

As a consequence of this, we show that $M_g$ does not retract onto $C$. Suppose we have a retraction $r : M_g \to C$. Then $r|_{M'_h} : M'_h \to C$ is also a retraction, which we showed cannot exist. Thus $M_g$ does not retract to $C$. □

**Proposition 0.7** (Exercise 1.2.9, part two)**.** *Let $M_g$ be the orientable surface of genus $g$, and let $C'$ be a "non-separating" circle (depicted on page 53 of Hatcher). Then $M_g$ retracts to $C'$.*

*Proof.* We think of $M_g$ as a $4g$-sided polygon with side identifications going around clockwise

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_{2g}, b_{2g}, a_{2g}^{-1}, b_{2g}^{-1}$$

where $C'$ is the loop $a_1$. Then we make the identification $a_i = a_1$ and $b_i = b_1$. The quotient map of this identification is then a retract $M_g \to M_1$, that is, a retract to the torus.

Now we claim that the torus can be retracted to $S^1$. Viewing the torus as a square with edges $aba^{-1}b^{-1}$, we define a retraction $M_1 \to M_1$ by projecting every point straight up to the edge $a_1$. Continuity is clear everywhere except near the bottom edge $a_1^{-1}$. But the map is continuous there as well, since this map is equivalent to projecting down to $a_1^{-1}$, and that map is clearly continuous everywhere except perhaps near the top edge $a_1$. So this is a retract of $M_1$ to $a_1$. Finally, we take the composition of these retractions to get a retration $M_g$ to $a_1 = C'$. □

**Proposition 0.8.** *The Borromean rings cannot be split apart.*

*Proof.* Label the three rings $A, B, C$. If we regard $C$ as lying in the complement of $A \cup B$, then the question of whether the three circles can be unlinked is equivalent to $C$ being trivial in $\pi_1(\mathbb{R}^3 \setminus (A \cup B))$. As noted by Hatcher on page 46, the complement of $A \cup B$ deformation retracts to $S^1 \vee S^1 \vee S^2 \vee S^2$. Then we see that $\pi_1(\mathbb{R}^3 \setminus (A \cup B))$ is the free group generated by loops around the two copies of $S^1$, which we can call $a$ and $b$. As Hatcher depicts on page 23, the loop $C$ is then $aba^{-1}b^{-1}$, which is not zero in the free group. Hence the rings cannot be split. $\square$